I Semester M.Sc. Mathematics Examination, January/February 2021
ALGEBRA

Time: 3 Hours **Max. Marks:80**

Instructions: Answer any five full questions. All questions carry equal marks.

1. a) Let H and K be subgroups of a group G. Then prove that HK is a subgroup of G, if and only if, $HK = KH$.
   b) State and prove Lagrange theorem for finite groups.
   c) Let $f: G \to G'$ be an onto group homomorphism with $Ker\ f = K$. Then show that $G' \cong G/K$.
   (6+5+5)
2. a) State and prove the First isomorphism theorem.
   b) State and prove the Cayley's theorem for permutations.
   c) Prove that for $n \geq 2$, the map $\varepsilon: S_n \to \{1, -1\}$ defined by signature is a homomorphism of the group S_n onto the multiplicative group of two elements $\{1, -1\}$. **(5+6+5)**
3. a) Prove that any group of order p^n, where p is prime, has nontrivial center.
   b) State and prove Second Sylow theorem.
   c) Show that a group of order $13^2 \cdot 17^2$ is abelian. **(6+6+4)**
4. a) Prove that every element x of F can be expressed as $x = ab^{-1}, a, b \in \mathcal{R}, b \neq 0$.
   b) Let $\mathcal{R}$ be a commutative ring with identity. Then show that $\mathcal{R}$ is a field if and only if the only ideals of $\mathcal{R}$ are $\{0\}$ and $\mathcal{R}$ itself.
   c) State and prove the correspondence theorem for ring homomorphism. **(4+6+6)**
5. a) Prove that P is a prime ideal of $\mathbb{Z}$ if and only if $P = (0)$ or $P = (p)$ for some prime p.
   b) If $\mathcal{R}$ is a commutative ring with identity and an integral domain, then prove that $\mathcal{R}[X]$ is also an integral domain.
   c) If p is a prime number such that $p \equiv 1(mod 4)$, then prove that p is a sum of two squares of integers. **(5+6+5)**
6. a) Prove that in a Principal Ideal Domain, a non-zero element is a prime if and only if it is irreducible and every non-zero prime ideal is maximal.
   b) State and prove the Factor theorem.
   c) Let $f(X) \in F[X]$ be of degree 2 or 3. Then prove that $f(X)$ is irreducible if and only if $f(X)$ has a root in F. **(6+6+4)**
7. a) If a vector space is generated by a finite set S, then prove that some subset of S is a basis for V and has a finite set.
   b) If K is algebraic over F and L is algebraic over K, then prove that L is algebraic over F.
   c) Find the splitting field of $x^2 - 2$ over Q. **(6+6+4)**
8. a) Any field of Characteristic zero is perfect and if $ch\ F = p$, then prove that F is perfect if and only if every element in F has a p^{th} root in F.
   b) State and prove the primitive element theorem. **(8+8)**

I Semester M.Sc. Mathematics Examination, January/February 2021

REAL ANALYSIS - I

Time: 3 Hours **Max. Marks: 80**

Instructions: Answer any five full questions. All questions carry equal marks.

1. **a)** Define an irrational number with an example. Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.
b) State and prove Cauchy Schwarz inequality.
c) Prove that for every real $x > 0$ and every integer $n > 0$, there is one and only one real $y > 0$ such that $y^n = x$. **(6+4+6)**

2. **a)** Define countable set. Prove that every subset of a countable set is countable.
b) Prove that the set all rational numbers is countable.
c) Let A be the set of all sequences whose elements are the digits 0 and 1. Then prove that A is uncountable. **(6+4+6)**

3. **a)** Define metric space. Prove that a set E in a metric space X is open if and only if its complement is closed.
b) Let X be a metric space and $E \subset X$. Then prove that,
i. $\bar{E}$ is closed.
ii. $E = \bar{E}$ if and only if E is closed.
iii. $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$. **(8+8)**

4. **a)** If a sequence $\{x_n\}$ converges then prove that its limit is unique.
b) Prove that a convergent sequence is bounded. Prove that converse is not true in general.
c) If $\{x_n\}$ is bounded and $\{y_n\}$ diverges to $+\infty$, then show that $\{x_n + y_n\}$ diverges to $+\infty$. **(5+5+6)**

5. **a)** Prove that a monotonically increasing sequence that is not bounded above diverges to $+\infty$. Also prove that a monotonically decreasing sequence that is not bounded below diverges to $-\infty$.
b) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $x_n \leq y_n$ for all n. Then prove that,
i. $\lim_{n\to\infty} \sup x_n \leq \lim_{n\to\infty} \sup y_n$
ii. $\lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \inf y_n$. **(8+8)**

6. **a)** Prove that if a sequence of real numbers is convergent then it is a Cauchy's sequence.
b) State and prove Cauchy's first limit theorem.
c) If $0 \leq x < 1$, show that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Further, if $\boldsymbol{x} \geq 1$ then prove that the series diverges.
(4+8+4)

7. **a)** Examine whether the series given are convergent.
i. $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n-1}$
ii. $\sum_{n=1}^{\infty} \frac{2n+1}{3n+2}$
b) State and prove Kummer's test and hence deduce ratio test. **(6+10)**

8. **a)** State and prove Dirichlet's test.
b) State and prove Abel's test. **(6+10)**

I Semester M.Sc. Mathematics Examination, January/February 2021

COMPLEX ANALYSIS - I

Time: 3 Hours **Max. Marks: 80**

Instructions: Answer any five full questions. All questions carry equal marks.

1. a) State and prove Lagrange's Identity

b) Let z be any complex number, then show that $Re\ z = \frac{z+\bar{z}}{2}$ and $Im\ z = \frac{z-\bar{z}}{2i}$.

c) Let z_1 and z_2 be the complex numbers, prove that

i. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

ii. $|z_1| = \overline{|z_1|}$ **(8+4+4)**

2. a) Prove that the points z_1, z_2 will be inverse points with respect to the circle

$$z_1\overline{z_2} + \bar{\alpha}z_1 + \alpha\overline{z_2} + r = 0$$

b) Find the distance between two points on Reimann sphere. **(8+8)**

3. a) State and prove necessary and sufficient conditions for a function to be analytic.

b) Prove that $f(z) = sinx \cdot\ coshy + i \cdot cosx\ \cdot sinhy$ is differentiable at every point. **(12+4)**

4. a) If $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$ then prove that,

i. $\lim_{z\to z_0}[f(z) + g(z)] = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z) = A + B.$

ii. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \frac{A}{B}, if\ B \neq 0.$

b) Find the general solution of a straight line.

c) Prove that the centroid of the triangle whose vertices are z_1, z_2, z_3 is $\frac{z_1 + z_2 + z_3}{3}$. **(6+6+4)**

5. a) Find the analytic function $f(z) = u + iv$ given that $u - v = e^x(\cos y - \sin y)$

b) State and prove necessary and sufficient condition required for sequence to be convergent.

c) State and prove Weirstrass m-test. **(4+6+6)**

6. a) Show that the inversion $w = \frac{1}{z}$ maps the circle $|z - 3| = 5$ to the circle $\left|w + \frac{3}{16}\right| = \frac{5}{16}$.

b) Find the bilinear transformation which maps the points $z = -1, 1, \infty$ respectively on $w = -i, -1, i$.

c) Prove that bilinear transformation preserves cross ratio. **(5+5+6)**

7. a) State and prove fundamental theorem of algebra.

b) Evaluate $\int_c \frac{z^3-2z+1}{(z-i)^2} dz$, where c is the circle $|z| = 2$.

c) State and prove Cauchy's theorem for a disc. **(6+4+6)**

8. a) State and prove maximum modulus theorem.

b) State and prove Taylor's theorem. **(8+8)**

I Semester M.Sc. Mathematics Examination, January/February 2021
DISCRETE MATHEMATICS

Time: 3 Hours **Max. Marks: 80**

Instructions: Answer any five full questions. All questions carry equal marks.

1. **a)** Define logical connectives and briefly explain all the logical connectives by writing their truth table.
b) Define Principle Disjunctive Normal Form (PDNF), Principle Conjunctive Normal Form (PCNF).
i) Obtain PDNF of $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$.
ii) Obtain PCNF of $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$ **(8+8)**

2. **a)** Briefly explain quantifiers, universal quantifiers and existential quantifiers.
b) Define negation of quantified statement. Then negate and simplify the following quantified statement.
i. $\exists x,\ [p(x) \vee q(x)]$
ii. $\forall x,\ [p(x) \rightarrow \sim q(x)]$ **(8+8)**

3. **a)** Establish the following results by using the Principles of Mathematical induction.
i. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
ii. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

b) State and prove extended pigeonhole principle. Then prove by using extended pigeonhole principle that in any set of 29 persons atleast five persons must have been born on the same day of the week. **(8+8)**

4. **a)** Define Catalan numbers. Using the move $R : (x, y) \rightarrow (x + 1, y)$ and $U : (x, y) \rightarrow (x, y + 1)$, find how many ways can one go.
i. from $(0,0)$ to $(6,6)$ and not above $y = x$.
ii. from $(2,1)$ to $(7,6)$ and not above $y = x - 1$.

b) i) Using generating function, find $P(6)$.
ii) Using generating function, find $P(6)$ into distinct summand. **(8+8)**

5. **a)** i) Solve the recurrence relation, $a_n + 4a_{n-1} + 4a_{n-2} = 8$ given $a_0 = 1, a_1 = 2$.
ii) Solve the recurrence relation using difference method , $y_{n+2} - 3y_{n+1} + 2y_n = 5^n + 2^n$.
b) Solve the Fibonacci recurrence relation, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$ given $F_0 = 0, F_1 = 1$.

(8+8)

6. **a)** State and prove the towers of Hanoi problem.
b) Explain generating function, mention its types and briefly explain them. **(8+8)**

7. **a)** Define equivalence relation. Prove that, if $R = \{(x, y); x - y\ is\ divisible\ by\ 3\}$ then R is an equivalence relation on the set of integers $\mathbb{Z}$.
b) Explain transitive closure. Prove that, if R is a relation on a set A then R^∞ is the transitive closure of R. **(8+8)**

8. **a)** Describe the procedure of Warshall's Algorithm
b) Define partial ordered set. Draw the Hasse diagram representing the positive divisors of 36. **(8+8)**

I Semester M.Sc. Mathematics Examination, January/February 2021
DIFFERENTIAL EQUATIONS

Time: 3 Hours **Max. Marks: 80**

Instructions: Answer any five full questions. All questions carry equal marks.

1. **a)** State the Lipschitz condition for a function f defined on closed domain D of the $xy-$ plane. Give an example of a function

i. which satisfy Lipschitz condition and

ii. Which do not satisfy Lipschitz condition. Justify your answer.

b) If $\phi_1(x)$ is a solution of $y'' + a_1(x)y' + a_2(x)y = 0$, then show that $\phi_2(x) = \phi_1(x)f(x)$ is a solution of this equation provided $f'(x)$ satisfies the equation $(\phi_1^2 y)' + a_1(x)\phi_1^2 y = 0$. Further show that $\phi_1(x)$ and $\phi_2(x)$ are linearly independent. **(8+8)**

2. **a)** State and prove Sturm's separation theorem.

b) Show that the differential equation $y'' + \frac{k}{x^2}y = 0,\ (X \le x < \infty)$, where k is a constant and $X > 0$, is oscillatory if $k > \frac{1}{4}$ and non-oscillatory if $k \le \frac{1}{4}$. **(10+6)**

3. **a)** Find eigen values and eigen functions of Sturm Liouvelli problem $y'' + \lambda y = 0,\ y(0) = y(\pi) = 0$.

b) Prove that the eigen values of a self-adjoint eigen value problem are real.

c) Establish Eigen function expansion formula. **(6+5+5)**

4. **a)** Find the series solution of the equation $y'' + xy' + (x^2 + 2)y = 0$ above the ordinary point $x = 0$.

b) Write the general form of Laguerre differential equation. Discuss its series solution and define its polynomials. **(8+8)**

5. **a)** Prove the following relations;

i. $L'_n(x) = L'_{n-1}(x) - L_{n-1}(x)$

ii. $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$

b) Find the general solution of following homogeneous linear equation

$$\frac{dX}{dt} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} X$$

(6+10)

6. **a)** Find all the critical points, their nature and stability of each of the critical points of the system

$$\frac{dx}{dt} = 8x - y^2; \qquad \frac{dy}{dt} = 6x^2 - 6y.$$

b) Define the following

i. Linear PDE

ii. Semi-linear PDE

iii. Quasi-linear PDE

iv. Non-linear PDE

c) Explain well-posed and ill-posed problems. **(8+4+4)**

7. **a)** Solve the Cauchy problem by the method of characteristics $u_x + u_y + u = 1$ with the initial data $u = \sin x$ on $y = x + x^2$.

b) Define complete integral of a non-linear first order partial differential equation. Find the complete integral of the equation $p^2 + qy - u = 0$. **(8+8)**

8. **a)** Explain the canonical transformation of solving second order partial differential equation.

b) Classify the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial y \partial x}$ and reduce it to canonical form. **(8+8)**

****